Arithmetic on Random Variables: Squeezing the Envelopes with New Joint Distribution Constraints

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Abstract

Uncertainty is a key issue in decision analysis and other kinds of applications. Researchers have developed a number of approaches to address computations on uncertain quantities. When doing arithmetic operations on random variables, an important question has to be considered: the dependency relationships among the variables. In practice, we often have partial information about the dependency relationship between two random variables. This information may result from experience or system requirements. We can use this information to improve bounds on the cumulative distributions of random variables derived from the marginals whose dependency is partially known.

Keywords. Uncertainty, arithmetic on random variables, distribution envelope determination (DEnv), joint distribution, dependency relationship, copulas, probability boxes, linear programming, partial information.

1 Introduction

Uncertainty is a key issue in decision analysis and other kinds of reasoning. Researchers have developed a number of approaches to address computations on uncertain distributions. Some of these approaches are confidence limits (Kolmogoroff 1941), discrete convolutions (i.e. Cartesian products, Ingram 1968), probabilistic arithmetic (Williamson and Downs 1990), Monte Carlo simulation (Ferson 1996), copulas (Nelsen 1999), stochastic dominance (Levy 1999), clouds (Neumaier 2004), and Distribution Envelope Determination (Berleant and Zhang 2004a).

Belief and plausibility curves, upper and lower previsions, left and right envelopes, and probability boxes designate an important type of representation for bounded uncertainty about distributions. When doing arithmetic operations on random variables that can result in such CDF bounds, an important question has to be considered: the dependency relationships among the variables. Couso et al. (1999) and Fetz and Oberguggenberger (2004) addressed different concepts of independence and their effects on CDF bounds. The copula-based approach can represent many interesting constraints on joint distributions that affect CDF bounds (e.g. Clemen 1999, Embrechts et al. 2003, Ferson and Burgman 1995). The copula-based approach is implemented as the part of RAMAS system (Ferson 2002). The Distribution Envelope Determination (DEnv) method can use Pearson correlation between marginals *X* and *Y* to squeeze CDF bounds of random variables derived from these marginals (Berleant and Zhang 2004b). This paper explores some additional constraints on dependency.

In practice, we may have partial information about the dependency relationship between two random variables. This information may result from empirical experience or system requirements. We can use this information to affect bounds on the cumulative distributions of new random variables derived from those whose dependency is partially known.

We focus on the following kinds of partial information.

1. Knowledge about probabilities of specified areas of the joint distribution of the marginals.
2. Knowledge about probabilities of specified ranges of values of the derived random variable.
3. Known relationships (>, <, =) among the probabilities of different areas of the joint distribution of the marginals.
4. Known relationships (>, <, =) among the probabilities of different ranges of the derived random variable.

Our method uses the DEnv algorithm (Berleant and Zhang 2004c).

2 Review of the Distribution Envelope Determination (DEnv) Algorithm

In this section, DEnv is reviewed briefly and abstractly, following Berleant and Zhang (2004a).

Suppose we have two samples x and y of random variables X and Y with probability density functions $f_x(.)$ and $f_y(.)$. Given a function g, a sample $z=g(x,y)$ of random variable Z is derived from x and y. DEnv is used to get the distribution of the derived variable Z. First, the input PDFs $f_x(.)$ and $f_y(.)$ are discretized by partitioning the support (i.e. the domain over which a PDF is non-zero) of each, yielding intervals $\mathbf{x}_i$, i=1…m, and $\mathbf{y}_j$, j=1…n. Each $\mathbf{x}_i$ is assigned a probability $p_{\mathbf{x}_i} = p(x \in \mathbf{x}_i) = \int_{x_0=\underline{\mathbf{x}_i}}^{\overline{\mathbf{x}_i}} f_x(x_0)dx_0$, where interval-valued symbols are shown in bold, and interval $\mathbf{x}_i$ has lower bound $\underline{\mathbf{x}_i}$ and upper bound $\overline{\mathbf{x}_i}$. Similarly, each $\mathbf{y}_j$ is assigned a probability $p_{\mathbf{y}_j} = p(y \in \mathbf{y}_j) = \int_{y_0=\underline{\mathbf{y}_j}}^{\overline{\mathbf{y}_j}} f_y(y_0)dy_0$. The $\mathbf{x}_i$'s and $\mathbf{y}_j$'s and their probabilities form the marginals of a discretized joint distribution called a joint distribution tableau (Table 1), the interior cells of which each contain two items. One is a probability mass $p_{ij} = p(x \in \mathbf{x}_i \wedge y \in \mathbf{y}_j)$. If x and y are independent then $p_{ij} = p(x \in \mathbf{x}_i) \cdot p(y \in \mathbf{y}_j) = p_{\mathbf{x}_i} \cdot p_{\mathbf{y}_j}$, where $p_{\mathbf{x}_i}$ is defined as $p(x \in \mathbf{x}_i)$ and $p_{\mathbf{y}_j}$ as $p(y \in \mathbf{y}_j)$. The second item is an interval that bounds the values $z=g(x,y)$ may have, given that $x \in \mathbf{x}_i \wedge y \in \mathbf{y}_j$. In other words, $\mathbf{z}_{ij}=\mathbf{g}(\mathbf{x}_i, \mathbf{y}_j)$.

$y\downarrow$ $x\rightarrow$ $z=g(x,y)$ ↘	…	$\mathbf{x}_i$ $p_{\mathbf{x}_i} = p(x \in \mathbf{x}_i)$	…
⋮	⋱	⋮	⋰
$\mathbf{y}_j$ $p_{\mathbf{y}_j} =$ $p(y \in \mathbf{y}_j)$	…	$\mathbf{z}_{ij}=\mathbf{g}(\mathbf{x}_i,\mathbf{y}_j)$ $p_{ij} = p(x \in \mathbf{x}_i \wedge y \in \mathbf{y}_j)$	…
⋮	⋰	⋮	⋱

Table 1: General form of a joint distribution tableau for random variables X and Y.

To better characterize the CDF $F_z(.)$, we next convert the set of interior cells of the joint distribution tableau into cumulative form. Because the distribution of each probability mass p_{ij} over its interval $\mathbf{z}_{ij}$ is not defined by the tableau, values of $F_z(.)$ cannot be computed precisely. However they can be bounded. DEnv does this by computing the analogous interval-valued function $\mathbf{F}_z(.)$ as

$$\underline{\mathbf{F}_z}(z_0) = \sum_{i,j|\overline{\mathbf{z}_{ij}} \le z_0} p_{ij} \text{ and } \overline{\mathbf{F}_z}(z_0) = \sum_{i,j|\underline{\mathbf{z}_{ij}} \le z_0} p_{ij}, \qquad (1)$$

resulting in right and left envelopes respectively bounding $F_z(.)$.

An additional complication occurs if the dependency relationship between x and y is unknown. Then the values of the p_{ij}'s are underdetermined, so equations (1) cannot be evaluated. However, the p_{ij}'s in column i of a joint distribution tableau must sum to $p_{\mathbf{x}_i}$ and the p_{ij}'s in row j must sum to $p_{\mathbf{y}_j}$ giving three sets of constraints: $p_{\mathbf{x}_i} = \sum_j p_{ij}$, $p_{\mathbf{y}_j} = \sum_i p_{ij}$, and $p_{ij} \ge 0$, for i=1…m, j=1…n. These constraints are all linear, and so may be optimized by linear programming. Linear programming takes as input linear constraints on variables, which in this case are the p_{ij}'s, and an expression in those variables to minimize, for example, $\underline{\mathbf{F}_z}(z_0) = \sum_{i,j|\overline{\mathbf{z}_{ij}} \le z_0} p_{ij}$ in equations (1) for some given value z_0. The output produced is the minimum value possible for $\underline{\mathbf{F}_z}(z_0)$, such that the values assigned to the p_{ij}'s are consistent with the constraints. $\overline{\mathbf{F}_z}(z_0) = \sum_{i,j|\underline{\mathbf{z}_{ij}} \le z_0} p_{ij}$ in equations (1) is maximized similarly. These envelopes are less restrictive (i.e. are farther apart) than when the p_{ij}'s are fully determined by an assumption of independence or some other given dependency relationship (in which case linear programming would not be needed).

These ideas could be generalized to n marginals, which would require an n-dimensional joint distribution tableau.

Next, we examine additional constraints that can be used to try to squeeze the envelopes closer together.

3 Knowledge about probabilities over specified areas of the joint distribution

Suppose we have information about the probabilities over given portions of the joint distribution. It could be that we know the probabilities exactly or perhaps we only know bounds on these values.

This problem breaks down into two major situations:

- ***Single-cell constraints***, where the probability of one p_{ij} is known in a joint distribution tableau, section 3.1.

- ***Multiple-cell constraints***, where our knowledge about probability spans more than one p_{ij}, section 3.2.

For ***multiple-cell constraints***, there are two subcategories:

- ***Area specified***, where we have knowledge about a sum $p_{ij}+\ldots+p_{mn}$, section 3.2.1.
- ***Probability of a function of the marginals specified over part of its domain***, where we have knowledge abut the probability of $g(x,y)$ over some interval $k_1 \le g(x,y) \le k_2$, section 3.2.2.

We explore these situations in the following examples. Assume that the marginal distributions of X and Y are known, and define $Z=X+Y$ as in Table 2.

$x\rightarrow$ $y\downarrow$ $z=x+y$	$\mathbf{x}_1=[x_{1l}, x_{1h}]$ $p_{\mathbf{x}_i}$	…	$\mathbf{x}_m=[x_{ml}, x_{mh}]$ $p_{\mathbf{x}_n}$
$\mathbf{y}_1=[y_{1l}, y_{1h}]$ $p_{\mathbf{y}_i}$	$\mathbf{z}_{11}=[x_{1l}+y_{1l}, x_{1h}+y_{1h}]$ p_{11}	…	$\mathbf{z}_{1m}=[x_{ml}+y_{1l}, x_{mh}+y_{1h}]$ p_{1m}
…	…	…	…
$\mathbf{y}_n=[y_{nl}, y_{nh}]$ $p_{\mathbf{y}_m}$	$\mathbf{z}_{1n}=[x_{1l}+y_{nl}, x_{1h}+y_{nh}]$ p_{n1}	…	$\mathbf{z}_{mn}=[x_{ml}+y_{nl}, x_{mh}+y_{nh}]$ p_{mn}

Table 2: Joint distribution tableau for the marginals X and Y, where $Z=X+Y$. Interval $\mathbf{x}_1$ has low bound x_{1l} and high bound x_{1h}, and similarly for other intervals.

Note the following row constraints:

$$\sum_{i=1}^{m} p_{ij} = p_{y_j} \quad \text{for } j=1 \text{ to } n, \tag{2}$$

and the following column constraints:

$$\sum_{j=1}^{n} p_{ij} = p_{x_i} \quad \text{for } i=1 \text{ to } m. \tag{3}$$

These are due to the properties of joint distributions.

The p_{ij}'s, i=1 to m, j=1 to n, are unknown. However, the row and column constraints limit the freedom of the p_{ij}'s significantly. This fact limits the space of feasible solutions for the linear programming problems in the DEnv algorithm. If we can get additional constraints, this space may be limited even more. That means that we could get bigger values for the minimization questions and/or smaller values for the maximization questions than we otherwise would obtain. Recall that in DEnv, the minimization values provide the right envelope and the maximization values provide the left envelope. If minimization outcomes become bigger or maximization outcomes become smaller, the left and right envelopes will become closer to each other. Thus we will get a more tightly specified space of possible CDFs for random variable Z, where Z is a function of the marginals.

Based on the example of Table 2, we demonstrate the use of constraints resulting from (1) ***single-cell constraints***, (2) ***multiple-cell constraints with area specified***, and (3) ***probability of a function of the marginals specified over part of its domain***, in the following subsections.

3.1 Single-cell constraints

Consider internal cells $\mathbf{z}_{ij}$ (Table 3). If only the row and column constraints hold, the probability p_{ij} of a given cell $\mathbf{z}_{ij}$ is not fully specified, but only constrained to some degree. Let us specify an additional stronger constraint on some p_{ij}, that it has some value $p_{ij}=c_{ij}$. This new constraint can be combined with the row and column constraints. This will tend to squeeze envelopes of Z closer together due to the general observation that more constraints tend to produce stronger conclusions.

This situation is relatively strict. To weaken it, the user may specify an inequality for p_{ij} such as $p_{ij} < c_{ij}$ or $\underline{\mathbf{c}}_{ij} \le p_{ij} \le \overline{\mathbf{c}}_{ij}$.

Here is an example. Consider two random variables X and Y having the discretized distribution shown in the joint distribution tableau of Table 3. $Z=X+Y$ is the derived random variable.

$x\rightarrow$ $y\downarrow$ $z=x+y$	$\mathbf{x}_1=[0,1]$ $p_{x_1}=0.2$	$\mathbf{x}_2=[3,4]$ $p_{x_2}=0.4$	$\mathbf{x}_3=[5,6]$ $p_{x3}=0.4$
$\mathbf{y}_1=[0,1]$ $p_{y_1}=0.4$	$\mathbf{z}_{11}=[0,2]$ p_{11}	$\mathbf{z}_{12}=[3,5]$ p_{12}	$\mathbf{z}_{13}=[5,7]$ p_{13}
$\mathbf{y}_2=[3,4]$ $p_{y_2}=0.6$	$\mathbf{z}_{21}=[3,5]$ p_{21}	$\mathbf{z}_{22}=[6,8]$ p_{22}	$\mathbf{z}_{23}=[8,10]$ p_{23}

Table 3: A joint distribution tableau for $Z=X+Y$.

Figures 1 & 2 show the CDFs of marginals X and Y implied by Table 3.

Suppose p_{11}= 0.16 is given (a single-cell constraint). If it is included with the original set of row and column constraints, the envelopes will tend to be squeezed together.

The sum of X and Y *without* the single-cell constraint p_{11}=0.16 is shown in Figure 3, while the sum *with* the constraint p_{11}=0.16 is shown in Figure 4. It is clear that the envelopes for $Z=X+Y$ are significantly narrowed as a result of this new constraint. If a weaker single-cell constraint is substituted for p_{11}=0.16, the envelopes are likely to be narrower than those of Figure 3, but wider than those of Figure 4. For example, Figure 5 shows the envelopes resulting from the constraint $0.15 \leq p_{11} \leq 0.17$.

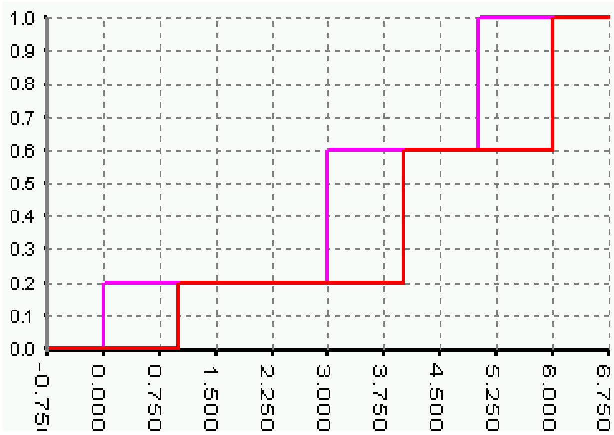

Figure 1: CDF envelopes for X.

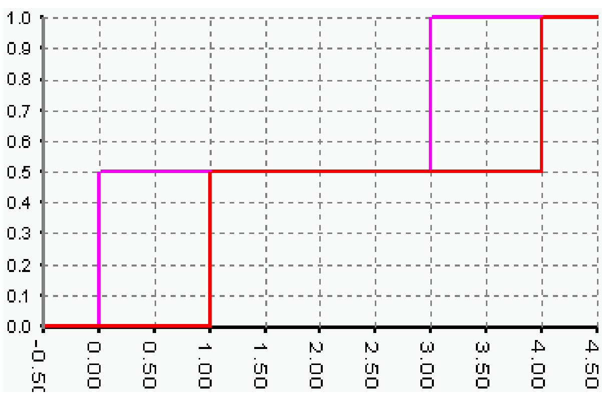

Figure 2: CDF envelopes for Y.

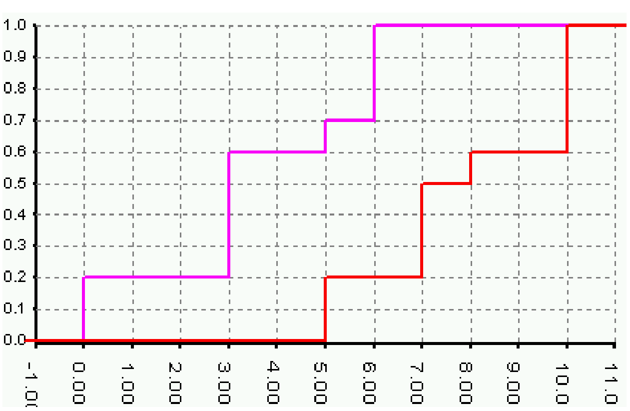

Figure 3: $\mathbf{F}_z(.)$ for $Z=X+Y$ without any extra constraints.

The envelopes shown in Figure 5 are closer together than those in Figure 3, but further apart than those in Figure 4.

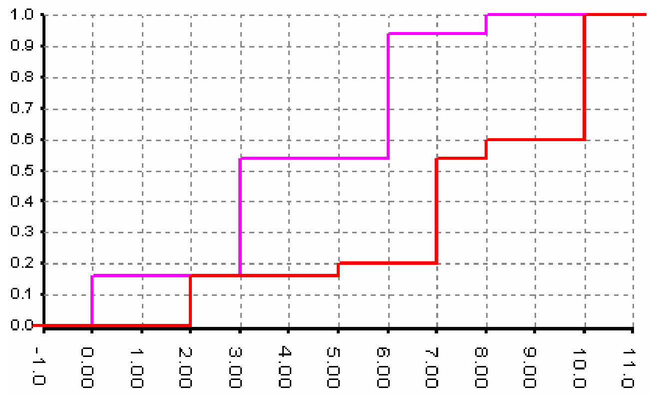

Figure 4: $\mathbf{F}_z(.)$ for $Z=X+Y$ with the single-cell constraint $p_{11}=0.16$.

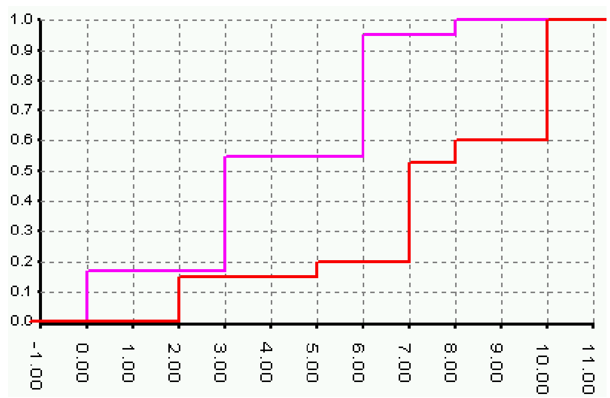

Figure 5: $\mathbf{F}_z(.)$ for $Z=X+Y$ with the single-cell constraint $0.15 \leq p_{11} \leq 0.17$.

3.2 Multiple-cell constraints

In section 3.1 we examined the situation where extra probabilistic information is available for *one* cell. This section explains the situation when extra probabilistic information is connected with a *set* of cells. This generalizes the case of the single-cell constraint. This situation includes two kinds of constraints: we will call these the ***area specified*** constraint and the ***probability of a function of the marginals specified over part of its domain*** constraint.

3.2.1 Area specified constraint

Here, a known probability describes the sum of the probabilities of multiple p_{ij}'s in the joint distribution tableau, instead of just one p_{ij}. This could occur if the probability of a certain region of the joint distribution is given, and that region spans multiple cells of the joint distribution tableau. However, the idea of constraining the probability of a summed probability of a number of cells generalizes to any set of cells, not just ones representing a contiguous region of the joint distribution.

For example, suppose $p_{11}+p_{12}+p_{21}=0.5$ in Table 3. Figure 6 shows the result of including this constraint with the row and column constraints of that table.

Compared with Figure 3, which has no extra constraints, this result has narrower envelopes.

3.2.2 Probability of a function of the marginals specified over part of its domain

Instead of focusing on the probability of areas of the joint distribution, as with the ***area specified*** constraint, this constraint focuses on probabilities of ranges of Z, where $z=g(x,y)$. To illustrate this situation, suppose that $p_z = p(z \in [0,5]) = 0.5$, where z is a sample of Z and $Z=X+Y$. The joint distribution tableau is as in Table 3. Then p_z *must* include p_{11}, p_{12}, and p_{21} because $\mathbf{z}_{11} = [0,2] \subset [0,5]$, $\mathbf{z}_{12} = [3,5] \subset [0,5]$, and $\mathbf{z}_{21} = [3,5] \subset [0,5]$. For all other $\mathbf{z}_{ij}$, $\mathbf{z}_{ij} \not\subset [0,5]$, so the probability of each such $\mathbf{z}_{ij}$ possibly could be distributed outside of [0,5], hence those $\mathbf{z}_{ij}$ might not contribute to p_z. Thus we have that $p_z = 0.5$ and $p_z \geq p_{11}+p_{12}+p_{21}$. This gives the constraint $0.5 \geq p_{11}+p_{12}+p_{21}$.

Figure 6: Results for $\mathbf{F}_z(.)$ using the area specified constraint of $p_{11}+p_{12}+p_{21}=0.5$.

Similarly, p_z *might* also include p_{13}. This would occur if $\mathbf{z}_{13}$ has its probability distributed as an impulse at its low bound of 5. This gives $0.5 \leq p_{11}+p_{12}+p_{21}+p_{13}$. These two constraints, $p_{11}+p_{12}+p_{21} \leq 0.5$ and $p_{11}+p_{12}+p_{21}+p_{13} \geq 0.5$, result from the given fact that $p_z = p(z \in [0,5]) = 0.5$. Figure 7 shows the results using these constraints.

The envelopes in Figure 7 are narrower than in Figure 3, due to the effects of the constraints used in the linear programming portion of the DEnv algorithm, $p_{11}+p_{12}+p_{21} \leq 0.5$ and $p_{11}+p_{12}+p_{21}+p_{13} \geq 0.5$, which are implied by the given fact. It is perhaps an interesting limitation of this approach that these constraints are weaker than the actual given, $p_z = p(z \in [0,5]) = 0.5$. Hence Figure 7, while an improvement over Figure 3, does not fully reflect the theoretical potential of the given to constrain the envelopes.

Figure 7: If $p_z = p(z \in [0,5]) = 0.5$, these envelopes result for $\mathbf{F}_z(.)$.

4. Known relationship among different areas of the joint distribution constraints

In the previous section we showed how probabilities of certain areas of a joint distribution can be used to narrow envelopes. In this section, we show how *relationships* among probabilities of different areas of the joint distribution can also be used to improve the CDF envelopes.

4.1 Unimodality constraint

If we know that the joint distribution is unimodal, this implies a set of relationships among different areas. For example, the fact that the probability density at the mode point is higher than it is in other areas implies constraints on the p_{ij}'s of Table 3. Define random variable Z as the sum of X and Y as in Table 2. The row and column constraints are in equations (2) & (3).

If we also know that X and Y have a unimodal joint distribution and that the mode point is in cell kl, the probability p_{kl} will be the result of a higher probability density than the other p_{ij}'s. Mathematically, $p_{kl} \geq p_{ij}$, $i \neq k$ and/or $j \neq l$, assuming the intervals $\mathbf{z}_{ij}$ have equal widths and do not overlap. If they do not have equal widths and/or they overlap, similar statements can be made that correct for the differences in widths and that take overlaps into account.

Now we have a set of new constraints. These constraints tend to decrease the area of the feasible solutions, narrowing the CDF envelopes.

Consider Table 3 again. If there is information about which cell $\mathbf{z}_{ij}$ contains the mode point, extra constraints may be derived. Suppose the mode point is in $\mathbf{z}_{23}$. Then the probability of p_{23} is greater than that of any other p_{ij}. Thus, $p_{23} \geq p_{ij}$, $i \neq 2$ or $j \neq 3$.

These constraints decrease the feasible solution range of original problem, enabling better envelopes to be obtained. Here are all the constraints including the new ones:

$$\sum_{i=1}^{2} p_{ij} = p_{y_j} \quad \text{for } j=1 \text{ to } 3,$$

$$\sum_{j=1}^{3} p_{ij} = p_{x_i} \quad \text{for } i=1 \text{ to } 2,$$

$p_{23} \geq p_{ij}$, $i \neq 2$ or $j \neq 3$.

The results using these constraints are depicted in Figure 8.

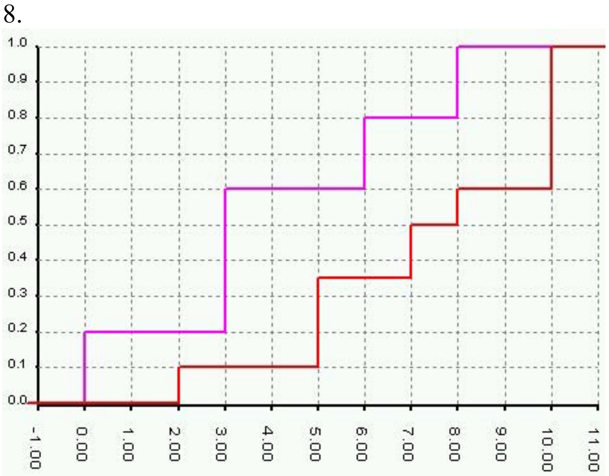

Figure 8: $\mathbf{F}_z(.)$, where the mode point is in $\mathbf{z}_{23}$.

Notice that the envelopes in Figure 8 are closer together than if the extra constraints are not present (as in Figure 3).

4.2 Conditional unimodality constraint

Here we examine another related, but somewhat different situation: conditional unimodality. In this situation, the joint distribution is known to be unimodal for x given a value for y, or unimodal for y given a value for x.

For example, suppose that given some value y of Y in $\mathbf{y}_2$=[3, 4] in Table 3, the maximum density of the PDF $f_x(x|y)$ is at some value of $x \in \mathbf{x}_3$. Then, the average probability density in the cell with probability p_{23} is greater than the average probability density in any cell with probability p_{2k}, $k \neq 3$. If the widths of intervals $\mathbf{z}_{2k}$ are the same, then $p_{23} \geq p_{2k}$, $k \neq 3$. In the more general case, the widths of the $\mathbf{z}_{2k}$ might not be the same. If width $w(\mathbf{z}_{23}) = c^*w(\mathbf{z}_{2k})$, then $p_{23} \geq c^*p_{2k}$. For the joint distribution tableau of Table 3, $w(\mathbf{z}_{21})=w(\mathbf{z}_{22})=w(\mathbf{z}_{23})$, so $p_{23} \geq p_{21}$ and $p_{23} \geq p_{22}$. These inequalities are constraints that, when included in the linear programming calls, will tend to squeeze the envelopes closer together than if these constraints were not included. Thus conditional unimodality can contribute constraints that tend to squeeze the envelopes bounding the CDF of Z.

Figure 9 shows the envelopes resulting from these new constraints. Notice that the envelopes are narrower than those of Figure 3, showing the narrowing influence of being able to assume conditional unimodality.

5. Results and Conclusion

In this paper, we present methods for using incomplete information about joint distributions to improve the envelopes around the CDF of a function of two marginals. More assumptions tend to give narrower result envelopes. More assumptions are good for improving results, but it is important that such assumptions are justified. We have shown that certain assumptions about the joint distribution of two marginals, that analysts will sometimes find useful and acceptable, can result in narrower CDF envelopes for functions of marginal random variables.

Figure 9: Conditional mode point in $\mathbf{z}_{23}$.

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